A DIFFERENTIAL GUIDANCE GAME FOR SYSTEMS WITH AFTEREFFECT

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The game problem of bringing controlled motions in a conflict situation onto a given set is considered for systems with aftereffect. The problem is investigated on the basis of the notion of extremal strategies previously introduced [1] for systems described by ordinary differential equations. The contents of the present study are related to those of [1-6].

1. Let us consider a system with aftereffect of the form

$$dx(t) / dt = f_1(t, x_t(s), u) + f_2(t, x_t(s), v) \qquad (1.1)$$

Here x is an n-dimensional phase vector; the r_1-dimensional vector u and the r_2-dimensional vector v are the controlling forces at the disposal of the first and second players, respectively. These forces are subject to the restrictions

$$u \in P, \qquad v \in Q \qquad (1.2)$$

where P and Q are compacts; the functionals $f_i(t, x(s), y)$ are defined on the products $[t_\alpha, t_\beta] \times C_{[-\tau,0]} \times Y_i$ $(Y_i = P, \quad Y_2 = Q)$, are continuous over all the arguments and satisfy the Lipschitz conditions in the functions $x(s)$

$$\| f_i(t, x_1(s), y) - f_i(t, x_2(s), y) \| \leqslant L \| x_1(s) - x_2(s) \|_\tau \qquad (1.3)$$

Here and below $C_{[-\tau,0]}$ is the space of continuous n-dimensional functions $x(s)$, $-\tau \leqslant s \leqslant 0$, $\tau = \text{const} \geqslant 0$, $L = \text{const} \geqslant 0$

$\| z \| = (z_1^2 + \ldots + z_m^2)^{1/2}$ is the norm in the Euclidean space E_m;
$\| x(s) \|_\tau = \max_s \| x(s) \|$ is the norm in $C_{[-\tau,0]}$;

the segment $x_t(s) = x(t+s)$ of the trajectory of system (1.1) is called the state of the system at the instant t (and is sometimes also denoted by the symbol $x_t(\cdot)$); the interval $[t_\alpha, t_\beta]$ contains all the time intervals over which the behavior of system (1.1) is considered.

The symbols and notations which appear below without references and explanations are all defined in [6]. The guidance problem to be considered is as follows.

Some closed set M is defined in the phase space of system (1.1). We are also given the initial position of the game, namely

$$p_0 = \{t_0, x_0(s)\} \qquad (t_0 \in [t_\alpha, t_\beta), \quad x_0(s) \in C_{[-\tau, 0]})$$

and the instant $\vartheta \in (t_0, t_\beta]$.

We are to construct the first-player strategy U which guarantees encounter of the motions $x[t, p_0, U, V_T]$ of system (1.1) with the target M at the given instant (by the given instant) ϑ. Here the motion $x[t, p_0, U, V_T]$ is assumed to be (see [6]) an n-dimensional vector function of the argument t which is constructed in the following way.

We take some covering Δ of the interval $[t_\alpha, t_\beta]$ by the half-intervals $[\tau_i, \tau_{i+1})$ $(\tau_0 = t_\alpha,\ i = 0, 1, \ldots)$ with the covering diameter $\delta = \sup_i (\tau_{i+1} - \tau_i) > 0$.

We denote by $x[t, p_0, U, V_T]_\Delta$ the absolutely continuous $(t \geqslant t_0)$ function $x[t]_\Delta$ which satisfies the condition $x[t_0 + s]_\Delta = x_0(s)$ and satisfies the contingency

$$\frac{dx[t]_\Delta}{dt} \in f_1(t, x_t[s]_\Delta, u[t]) + F_2(t, x_t[s]_\Delta) \tag{1.4}$$
$$u[t] = u[\tau_i] \in U(\tau_i, x_{\tau_i}[s]_\Delta), \qquad \tau_i \leqslant t < \tau_{i+1}$$

for almost all $t \in [t_0, t_\beta]$.

The sets $U(t, x(s))$ define the strategy U

$$F_2(p) = F_2(t, x(s)) = \overline{c0}\,\{f_2(t, x(s), v) \mid v \in Q\}$$

and the symbol $\overline{c0}\{z\}$ denotes the closure of the convex shell of the set of vectors z.

Then, by definition, $x[t, p_0, U, V_T]$ is a continuous function which has the following property: there exists a sequence of coverings $\{\Delta_j\}$ with $\{\delta_j\} \to 0$ such that some sequence of functions $\{x[t, p_0, U, V_T]_{\Delta j}\}$ converges in $C_{[t_0, t_\beta]}$ to $x[t, p_0, U, V_T]$.

We note that by virtue of the equiboundedness and equicontinuity of the set of solutions of the equation
$$\frac{dx(t)}{dt} \in F_1(t, x_t(s)) + F_2(t, x_t(s))$$
$(x(t_0 + s) = x_0(s);\ F_1(p) = F_1(t, x(s)) = \overline{c0}\{f_1(t, x(s), u) \mid u \in P\};\ t_0 \leqslant t \leqslant t_\beta)$ the set of motions $\{x[t, p_0, U, V_T]\}$ defined in this way is nonempty).

Let us refine our statement of the problem. Let $\rho(x, M)$ be the distance in E_n from the point x to the set M.

Definition 1.1. For a given initial game position p_0 the strategy U guarantees encounter of the motions $x[t] = x[t, p_0, U, V_T]$ of system (1.1) with the target M at the instant ϑ (by the instant ϑ) if

$$\rho(x[\vartheta], M) = 0 \qquad (\min_{t_0 \leqslant t \leqslant t_\beta} \rho(x[t], M) = 0) \tag{1.5}$$

where $x[t]$ is any motion $x[t, p_0, U, V_T]$.

The sufficient conditions of solvability of the guidance problem are given and the structure of the required strategy U is investigated below.

2. Let each $t \in [t_\alpha, t_\beta]$ be associated with a nonempty set $W_t = W_t\{x(s)\} \subset C_{[-\tau, 0]}$. We take a specific number $\xi \in [-\tau, 0]$ and call the set

$$W_{t\xi} = \{x(\xi) \mid x(s) \in W_t\}$$

the ξ-section of the set W_t. The sequence $\{x^{(k)}(\xi)\}$, where $x^{(k)}(s) \in C_{[-\tau,0]}$ will be called the ξ-section of the sequence $\{x^{(k)}(s)\}$.

We set
$$r(x(s), W_t) = \inf \| x(s) - y(s) \|_\tau \quad (y \in W_t) \tag{2.1}$$

Let $\{y\} = \{x^{(k)}(s)\}$ be some sequence which minimizes (2.1) for a given $x(s)$.

Let us construct the set of partial limits of the sequence $\{x^{(k)}(0)\}$ which is the 0-section of the sequence $\{x^{(k)}(s)\}$.

We denote by $Z(x(0))$ the collection of elements of this set which are closest to

$x(0)$ in E_n.

Definition 2.1. We define strategies extremal to the system of sets W_t, $t_0 \leqslant t \leqslant \vartheta$, as those strategies U^e, V^e which are defined by the sets $U^e(t, x(s))$, $V^e(t, x(s))$, respectively, constructed according to the rule

$$U^e(t, x(s)) = \{u_e \mid (z - x(0)) f_1(t, x(s), u_e) = \max (z - x(0)) f_1(t, x(s), u)\} \qquad (u \in P) \tag{2.2}$$

$$V^e(t, x(s)) = \{v_e \mid (z - x(0)) f_2(t, x(s), v_e) = \max (z - x(0)) f_2(t, x(s), v)\} \qquad (v \in Q)$$

for at least one $z \in Z(x(0))$.

Theorem 2.1. Let a system of strongly u-stable sets W_t, $t_0 \leqslant t \leqslant \vartheta$ (see [6]) be specified in the interval $[t_0, \vartheta]$, and let $M \supset W_{\vartheta 0}$. If the initial game position $p_0 = \{t_0, x_0(s)\}$ satisfies the condition $r(x_0, W_{t_0}) = 0$, then the first-player strategy U^e extremal to the system of sets W_t guarantees encounter of the motions $x[t] = x[t, p_0, U^e, V_T]$ of system (1.1) with the target M at the instant ϑ.

This theorem follows from the following lemma, which is also of independent interest.

Lemma 2.1. Let the initial game position $p_0 = \{t_0, x_0(s)\}$ be such that $r(x_0(s), W_{t_0}) = 0$. If the system of sets W_t, $t_0 \leqslant t \leqslant \vartheta$ be strongly u-stable [6], then the strategy U^e extremal to it satisfies the condition

$$r(x_t[s], W_t) = 0, \qquad t_0 \leqslant t \leqslant \vartheta \tag{2.3}$$

where $x[t]$ is any motion $x[t, p_0, U^e, V_T]$.

Proof. Let the system of sets W_t, $t_0 \leqslant t \leqslant \vartheta$, be strongly u-stable, and let $r(x_0(s), W_{t_0}) = 0$. Let $x[t]$ be an arbitrary motion from the collection $\{x[t, p_0, U^e, V_T]\}$.

By the definition of this motion there exists a sequence of functions

$$\{x[t]_{\Delta_j}\} = \{x[t, p_0, U^e, V_T]_{\Delta_j}\} \quad (c\{\delta_j\} \to 0),$$

which converges uniformly to $x[t]$ on $[t_0, \vartheta]$.

The validity of relation (2.3) is clearly established once we have shown that whatever the positive number ε_0, the segment $x_t[s]_{\Delta_j}$ of any function $x[t]_{\Delta_j}$ with a sufficiently large number j lies in the ε_0-neighborhood $W_t^{\varepsilon_0}$ of the set W_t for any $t \in (t_0, \vartheta]$.

To this end we choose from the sequence $\{x[t]_{\Delta_j}\}$ in arbitrary fashion a function $x[t]_\Delta$ and construct along it the estimate of the quantity $\varepsilon_\Delta[\tau_{i+1}]$ in terms of the quantities $\varepsilon_\Delta[\tau_i]$ and δ. Here and below $\varepsilon_\Delta[t] = r(x[t]_\Delta, W_t)$.

Let $z(\tau_i)_\Delta$ be an element of the set $Z(x_{\tau_i}[0]_\Delta)$ which for $t = \tau_i$ defines in accordance with (2.2) the control $u_e[t]$ corresponding to the extremal strategy U^e. Without limiting generality we assume that the section $\{x_{\tau_i}^{(k)}(0)_\Delta\}$ of the minimizing sequence $\{x_{\tau_i}^{(k)}(s)_\Delta\}$ which generates the vector $z(\tau_i)_\Delta$ converges to $z(\tau_i)_\Delta$. From (2.2) we have

$$(x_{\tau_i}[0]_\Delta - x_{\tau_i}^{(k)}(0)_\Delta) N_1(\tau_i, u) \leqslant \beta_1(k) \qquad (u \in P) \tag{2.4}$$

Here

$$N_1(\tau_i, u) = f_1(\tau_i, x_{\tau_i}[s]_\Delta, u_e) - f_1(\tau_i, x_{\tau_i}[s]_\Delta, u), \quad \beta_1(k) \to 0 \quad \text{for} \quad k \to \infty$$

Let us consider the position $p(k, i) = \{\tau_i, x_{\tau_i}^{(k)}(s)_\Delta\}$. By virtue of the strong u-stability of the system of sets W_t, $t_0 \leqslant t \leqslant \vartheta$, among the motions

$$x^{(k)}[t]_\Delta = x[t, p(k, i), U_T, V_{v_0}]$$

there exists a motion with the property

$$x^{(k)}_{\tau_{i+1}}[s] \in W_{\tau_{i+1}} \tag{2.5}$$

Here the strategy V_{v_0} is generated by the function

$$v_0(t) = v_0[\tau_i] = v_0, \qquad \tau_i \leqslant t \leqslant \tau_{i+1}$$

which satisfies the following condition for any $v \in Q$:

$$(x_{\tau_i}[0]_\Delta - z(\tau_i)_\Delta)\, N_2(\tau_i, v) \leqslant 0 \tag{2.6}$$

$$N_2(\tau_i, v) = f_2(\tau_i, x_{\tau_i}[s], v) - f_2(\tau_i, x_{\tau_i}[s]_\Delta, v_0)$$

and therefore the following condition (for any $v \in Q$):

$$(x_{\tau_i}[0]_\Delta - x^{(k)}_{\tau_i}(0)_\Delta)\, N_2(\tau_i, v) \leqslant \beta_2(k) \tag{2.7}$$

$$\beta_2(k) \to 0 \quad \text{as} \quad k \to \infty$$

By the definition of the quantity $\varepsilon_\Delta[t]$ with allowance for (2.5) we have the estimate

$$\varepsilon_\Delta[\tau_{i+1}] \leqslant \| x_{\tau_{i+1}}[s]_\Delta - x^{(k)}_{\tau_{i+1}}[s]_\Delta \|_\tau \tag{2.8}$$

We note, furthermore, that the segments $x_{\tau_{i+1}}[s]_\Delta$, $x^{(k)}_{\tau_{i+1}}[s]_\Delta$ of the trajectories $x[t]_\Delta$, $x^{(k)}[t]_\Delta$ can be expressed as follows (we assume that $\tau_{i+1}-\tau_i \leqslant \tau$):

$$x_{\tau_{i+1}}[s]_\Delta = x_{\tau_i}[0]_\Delta + \int_{\tau_i}^{\tau_{i+1}+s} \{f_1(t, x_t[\cdot]_\Delta, u_e) + \varphi_2[t]\}\, dt, \qquad -\alpha_i \leqslant s \leqslant 0$$

$$x_{\tau_{i+1}}[s]_\Delta = x_{\tau_i}[s+\alpha_i], \qquad -\tau \leqslant s \leqslant -\alpha_i \tag{2.9}$$

$$x^{(k)}_{\tau_{i+1}}[s]_\Delta = x^{(k)}_{\tau_i}(0)_\Delta + \int_{\tau_i}^{\tau_{i+1}+s} \{\varphi_1^{(k)}[t] + f_2(t, x_t^{(k)}[\cdot]_\Delta, v_0(t))\}\, dt$$

$$-\alpha_i \leqslant s \leqslant 0$$

$$x^{(k)}_{\tau_{i+1}}[s]_\Delta = x^{(k)}_{\tau_i}(s+\alpha_i), \qquad -\tau \leqslant s \leqslant \alpha_i \quad (\alpha_i = \tau_{i+1} - \tau_i)$$

Here $\varphi_1^{(k)}[t]$, $\varphi_2[t]$ are summable functions which satisfy the following inclusions for almost all $t \in [\tau_i, \tau_{i+1})$:

$$\varphi_1^{(k)}[t] \in F_1(t, x_t^{(k)}[s]_\Delta), \qquad \varphi_2[t] \in F_2(t, x_t[s]_\Delta)$$

By virtue of the definitions of the motions $x[t, p_0, U, V_T]$, $x[t, p_0, U_T, V_v]$ and relations (2.9), we obtain from (2.8):

$$\varepsilon_\Delta[\tau_{i+1}] \leqslant \max\{\max_{-\tau \leqslant s \leqslant -\alpha_i} \| x_{\tau_i}[s]_\Delta - x^{(k)}_{\tau_i}(s)_\Delta \|$$

$$\max_{-\alpha_i \leqslant s \leqslant 0} \| x_{\tau_i}[0]_\Delta - x^{(k)}_{\tau_i}(0)_\Delta + J_1(s) + J_2(s) \|\} \tag{2.10}$$

Here

$$J_1(s) = \int_{\tau_i}^{\tau_{i+1}+s} \{f_1(t, x_t[\cdot]_\Delta, u_e) - \varphi_1^{(k)}[t]\}\, dt$$

$$J_2(s) = \int_{\tau_i}^{\tau_{i+1}+s} \{\varphi_2[t] - f_2(t, x_t^{(k)}[\cdot]_\Delta, v_0(t))\}\, dt$$

Recalling the continuity of the sets $F_i(t, x(s))$ with respect to t, $x(s)$ and Lipschitz' condition (1.3), we find that

$$J_m(s) = (\alpha_i + s)(p_m + q_m) + \int\limits_{\tau_i}^{\tau_i+1+s} r_m^{(k)}(t)\,dt \qquad (2.11)$$

$$p_m \in \overline{c\,0}\,\{N_m(\tau_i,\ y) \mid y \in Y_m\}$$

$$\| r_m^{(k)} \| \leqslant L \| x_t[s]_\Delta - x_t^{(k)}[s]_\Delta \|_\tau \quad (m = 1,\ 2)$$

where $\| q_m \| \to 0$ as $\alpha_i \to 0$ uniformly in $\tau_i \in [t_0, \vartheta]$.

We shall now show that whatever the positive number β, all the functions $x\ [t]_{\Delta_j}$ with a sufficiently large number j satisfy the inequality

$$\varepsilon_{\Delta_j}[t] \leqslant \beta \exp\left[3L(t - t_0)\right] \qquad (2.12)$$

for all $t \in [t_0, \vartheta]$.

In fact, assuming that the opposite statement holds, we infer that there exists a number β_0 such that for any number j_0 there exists a number $j \geqslant j_0$ and an instant $t_*(j) \in [t_0, \vartheta]$ for which inequality (2.12) is violated for $\beta = \beta_0$. By the condition of the theorem, at the initial instant $t = t_0$ for any j we have $\varepsilon_{\Delta_j}[t_0] = 0$. Let us assume that at the points τ_k condition (2.12) for the functions $x\ [t]_{\Delta_j}$ is first violated for $t_*(j) = \tau_{i+1} = \tau_{i+1}(j)$,

$$\varepsilon_{\Delta_j}[\tau_{i+1}] > \beta_0 \exp\left[3L(\tau_{i+1} - t_0)\right] \qquad (2.13)$$

Then for $t = \tau_i = \tau_i(j)$ for the same functions we have

$$\varepsilon_{\Delta_j}[\tau_i] \leqslant \beta_0 \exp\left[3L(\tau_i - t_0)\right] \qquad (2.14)$$

Let us choose a positive number $\beta_1 \leqslant \beta_0$. For functions $x\ [t]_{\Delta_j}$ which satisfy conditions (2.13), (2.14) we have one of two cases:

Case 1. For any β_1 there exists a number $j(\beta_1)$ such that

$$\varepsilon_{\Delta_j}[\tau_i] < \beta_1 \qquad (2.15)$$

for $j \geqslant j(\beta_1)$.

Case 2. There exists a number β_1 such that for any number j_0 there exists a number $j \geqslant j_0$ such that

$$\varepsilon_{\Delta_j}[\tau_i] \geqslant \beta_1 \qquad (2.16)$$

In Case 1 expression (2.11) implies the estimate

$$\varepsilon_{\Delta_j}[\tau_{i+1}] \leqslant \beta_1 + O(j), \qquad (O(j) \to 0 \quad \text{as} \quad j \to \infty) \qquad (2.17)$$

For a sufficiently small β_1 and large j inequality (2.17) contradicts condition (2.13).

Let us consider Case 2. If for all functions $x\ [t]_{\Delta_j}$ with a sufficiently large number j we have the inequality

$$\| x_{\tau_i}[0]_{\Delta_j} - z(\tau_i)_{\Delta_j} \| < \alpha \qquad (2.18)$$

where α is an arbitrarily small positive number, then, choosing a sufficiently large k, we obtain the following estimate for these functions from relation (2.10):

$$\varepsilon_{\Delta_j}[\tau_{i+1}] \leqslant \| x_{\tau_i}[s]_{\Delta_j} - x_{\tau_i}^{(k)}(s)_{\Delta_j} \|_\tau \qquad (2.19)$$

This estimate implies the inequality

$$\varepsilon_{\Delta_j}[\tau_{i+1}] \leqslant \varepsilon_{\Delta_j}[\tau_i] \qquad (2.20)$$

If among the functions $x\ [t]_{\Delta_j}$ for which Case 2 holds there are functions with arbitrarily large numbers j (at a certain positive α) such that

$$\| x_{\tau_i}[0]_{\Delta_j} - z(\tau_i)_{\Delta_j} \| \geqslant \alpha \qquad (2.21)$$

then, substituting (2.11) into (2.10), choosing a sufficiently large k, and recalling (2.4), (2.7), we obtain the relation

$$\varepsilon_{\Delta_j}[\tau_{i+1}] \leqslant (1+2L\alpha_i)\|x_{\tau_i}[s]_{\Delta_j} - x_{\tau_i}^{(k)}(s)_{\Delta_j}\|_\tau + o(\alpha_i) \qquad (2.22)$$

Here $o(\alpha_i)$ has a higher order of smallness than α_i uniformly in k and $\tau_i \in [t_0, \vartheta]$. This implies the estimate

$$\varepsilon_{\Delta_j}[\tau_{i+1}] \leqslant (1+2L\delta_j)\,\varepsilon_{\Delta_j}[\tau_i] + o(\delta_j) \qquad (2.23)$$

$$\delta_j^{-1} o(\delta_j) \to 0 \quad \text{as} \quad j \to \infty \qquad (\text{uniformly in } \tau_i \in [t_0, \vartheta])$$

Relations (2.20), (2.21) clearly contradict the collection of inequalities (2.13), (2.14).

Thus, inequality (2.12) has been proved. This implies that all functions $x[t]_{\Delta_j}$ with a sufficiently large j satisfy the condition

$$\varepsilon_{\Delta_j}[t] \leqslant \varepsilon_0, \qquad t_0 \leqslant t \leqslant \vartheta \qquad (2.24)$$

where ε_0 is an arbitrary and arbitrarily small positive number. From (2.24) and the definition of the motion $x[t] = x(t, p_0, U, V_T]$ we infer relation (2.3).

The following statement also follows directly from the above reasoning.

Lemma 2.2. Let a system of strongly u-stable sets W_t, $t_0 \leqslant t \leqslant \vartheta$ be specified in the interval $[t_0, \vartheta]$. The strategy U^e extremal to this system of sets has the following property: whatever the positive number ε, there exists a positive number $\alpha = \alpha(\varepsilon)$ such that the following inequality is fulfilled for all motions $x[t] = x[t, p_0, U^e, V_T]$ of system (1.1):

$$r(x_t[s], W_t) < \varepsilon, \qquad t_0 \leqslant t \leqslant \vartheta$$

provided the initial game position $p_0 = \{t_0, x_0(s)\}$ satisfies the inclusion

$$x_0(s) \in W_{t_0}^{\alpha}.$$

Here W_t^α is the α-neighborhood in $C_{[-\tau,0]}$ of the set W_t, i.e. the collection of elements $x(s) \in C_{[-\tau,0]}$ of the form

$$x(s) = y(s) + z(s), \qquad y(s) \in W_t, \qquad \|z(s)\|_\tau \leqslant \alpha$$

Note 2.1. The extremal second-player strategy V^e has properties analogous to those of U^e. Specifically, the following statements hold.

Lemma 2.3. Let the initial game position $p_0 = \{t_0, x_0(s)\}$ be such that $r(x_0(s), W_{t_0}) = 0$. If the system of sets W_t, $t_0 \leqslant t \leqslant \vartheta$ is strongly v-stable (see [6]), then the strategy V^e extremal to it satisfies the condition

$$r(x_t[s], W_t) = 0, \qquad t_0 \leqslant t \leqslant \vartheta$$

where $x[t]$ is any motion $x[t, p_0, U_T, V^e]$ (see [6]).

Lemma 2.4. Let the system of sets W_t, $t_0 \leqslant t \leqslant \vartheta$ be strongly v-stable. For any positive number ε there exists a positive number $\alpha = \alpha(\varepsilon)$ such that the following inequality holds for all motions $x[t] = x[t, p_0, U_T, V^e]$ of system (1.1):

$$r(x_t[s], W_t) < \varepsilon, \quad t_0 \leqslant t \leqslant \vartheta$$

provided the initial game position $p_0 = \{t_0, x_0(s)\}$ satisfies the inclusion $x_0(s) \in W_{t_0}^\alpha$.

Now let us consider the problem of encounter of system (1.1) with the target M by the instant ϑ.

The following statement is valid.

Theorem 2.2. Let the initial game position $p_0 = \{t_0, x_0(s)\}$ be such that $r(x_0(s), W_{t_0}) = 0$. If the system of sets W_t, $t_0 \leqslant t \leqslant \vartheta$ is u-stable, and if $M \supset W_{\vartheta_0}$, then the strategy U^e extremal to this system guarantees encounter of the motions $x[t, p_0, U^e, V_T]$ of system (1.1) with the target M by the instant ϑ.

Proof. As before, let $x[t]$ be an arbitrary motion from the collection $\{x[t, p_0, U^e, V_T]\}$, and let $\{x[t]_{\Delta_j}\}$ be the sequence of functions $x[t]_\Delta = x[t, p_0, U^e, V_T]_\Delta$. To prove the statement of the theorem (see Definition 1.1) we need merely to verify that all the functions $x[t]_{\Delta_j}$ with a sufficiently large number j satisfy the inequality

$$\min_{t_0 \leqslant t \leqslant \vartheta} \rho(x[t]_{\Delta_j}, M) < \varepsilon \tag{2.25}$$

where ε is an arbitrarily small positive number.

Assuming the opposite, we find that there exists a positive number ε_0 such that for any number j_0 there exists a number $j \geqslant j_0$ for which

$$\min_{t_0 \leqslant t \leqslant \vartheta} \rho(x[t]_{\Delta_j}, M) \geqslant \varepsilon_0 \tag{2.26}$$

Let us consider the subsequence of functions $x[t]_{\Delta_j}$ each of whose terms satisfies condition (2.26). We denote this subsequence by $\{x[t]_{\Delta_j}\}$ as before. We now denote the i th node τ_i $(i = 0, 1, \ldots)$ of the decomposition of Δ_j by the symbol $\tau_i[j]$. As above, let

$$\{x^{(k)}_{\tau_i[j]}(s)\} \qquad (k = 1, 2, \ldots)$$

be a minimizing sequence for (2.1). where

$$x(s) = x_{\tau_i[j]}[s]_{\Delta_j}$$

Here the 0-section of this sequence $\{x^k_{\tau_i[j]}(0)\}$ converges to $z(\tau_i[j])_{\Delta_j}$ (see Sect. 2 above). Let

$$x^{(k)}[t; \tau_i[j]] = x^{(k)}[t, p(k, \tau_i), U_T, V_{v_0}], \quad p(k, i) = \{\tau_i[j], x^{(k)}_{\tau_i[j]}(s)\}$$

where the function v_0 satisfies (2.7) for $\Delta = \Delta_j$, and the motion $(0$ has the property (translator's note: there is obviously an omission in the original text at this point). The following inclusion is fulfilled:

$$x^{(k)}_{\tau_{i+1}[j]}[s, \tau_i[j]] \in W_{\tau_{i+1}[j]} \tag{2.27}$$

or the condition

$$x^{(k)}[t(j), \tau_i[j]] \in M \tag{2.28}$$

holds for at least one $t = t(j) \in [\tau_i[j], \tau_{i+1}[j])$.

Such a motion exists by virtue of the inclusion

$$x^{(k)}_{\tau_i[j]}(s) \in W_{\tau_i[j]}$$

and by virtue of the definition of the u-stability of the system of sets W_t, $t_0 \leqslant t \leqslant \vartheta$ (see [6]).

Two cases are possible for the functions $x[t]_{\Delta_j}$ from $\{x[t]_{\Delta_j}\}$:

Case 1. Either there exists a number j_* such that for any $j \geqslant j_*$ and any $\tau_i[j]$ there exists a number k_* such that inclusion (2.27) holds for any motion $x^{(k)}[t, \tau_i[j]]$ with $k \geqslant k_*$;

Case 2. Or for any number j^* there exists a number $j \geqslant j^*$ and a node $\tau_m[j]$ such that the collection $\{x^{(k)}[t, \tau_m[j]], k = 1, 2, \ldots\}$ contains motions with arbitrarily large numbers k for which condition (2.28) holds. But then choosing (if necessary) a subsequence from $\{x^{(k)}_{\tau_m[j]}(s)\}$, we can clearly assume that condition (2.28) for $x^{(k)}[t, \tau_m[j]]$ holds for all sufficiently large k.

Let Case 1 hold. Then (see the proof of Lemma 2.1) estimate (2.12) holds for the functions $x[t]_{\Delta_j}$. Making use of this estimate and recalling the inclusion $W_{\vartheta_0} \subset M$ and the inequality $\rho(x[t]_\Delta, M) \leqslant \varepsilon_\Delta[t]$, we find that for sufficiently large j we have

$\rho\,(x[\vartheta]_{\Delta_j}, M) \leqslant \varepsilon_0$, which contradicts (2.26).

Now let us consider Case 2. Without limiting generality, we assume that $[\tau_m[j], \tau_{m+1}[j])$ is the first half-interval for the function $x[t]_{\Delta_j}$, where condition (2.28) holds. It can be verified directly that Case 2 implies the following estimate for the functions $x[t]_{\Delta_j}$:

$$\rho\,(x[t(j)]_{\Delta_j}, M) \leqslant \varepsilon_{\Delta_j}[\tau_m[j]] + O(j) \qquad (2.29)$$

$O(j) \to 0$ as $j \to \infty$ (uniformly in $\tau_i \in [t_0, \vartheta]$).

Next, arguments similar to those used in proving Lemma 2.1 can be adduced to show that every function $x[t]_{\Delta_j}$ from Case 2 which has a sufficiently large number j satisfies inequality (2.12) (where β is an arbitrarily small positive number) in $[t_0, \tau_m[j]]$. But then (2.29) and (2.12) (for $t = \tau_m[j]$) imply that for sufficiently large j we have the relation $\rho\,(x[t]_{\Delta_j}, M) < \varepsilon_0$, which also contradicts condition (2.26). The theorem has been proved.

Note 2.2. Theorems 2.1 and 2.2 clearly remain valid if the set $M = M(t)$ depends continuously on t. In this case the condition $W_{\vartheta_0} \subset M$ in the statements of the theorems must be replaced by the inclusion $W_{\vartheta 0} \subset M(\vartheta)$.

Note 2.3. In connection with Theorems 2.1 and 2.2 there arises the question of the existence of a system of sets W_t, $t_0 \leqslant t \leqslant \vartheta$, having the required stability properties. This matter is discussed in [6], where the sufficient conditions of strong u-stability of program absorption of the target M by system (1.1) are indicated. This paper also states (without proof) that the system of positional absorption sets (see [6]) has the property of u-stability. This is particularly important (in connection with Theorem 2, 2) in solving the game problem on the minimax (maximin) of the time to encounter of system (1.1) with the target M (see [2]).

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